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Scaling theory for the statistics of self-avoiding walks on random lattices

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Abstract. We study self-avoiding walks (SAW) on randomly diluted (quenched) lattices with direct configurational averaging over the moments of the SAW distribution function. A scaling function representation of R_N, the average end-to-end distance of N-step walks, is studied here both for SAW on (a) the infinite percolation cluster and (b) any cluster. We have shown that, at the percolation threshold, if $R_N \sim N^{\nu^{P_c}}$ for the infinite cluster averaging and $R_N \sim N^{\tilde{\nu}^{P_c}}$ for all cluster averaging, then $\tilde{\nu}^{P_c} = \nu^{P_c}(1-\beta_P/2\nu_P)$, where β_P and ν_P are the percolation order parameter and correlation length exponents respectively. We also propose a scaling function representation for the total number of N-step SAW configuration $G_N(\sim \mu^N N^{\gamma-1})$ for infinite cluster averaging, which gives $\gamma^{P_c} = \gamma + d(\nu^{P_c} - \nu)$. For all cluster averaging γ will remain unchanged.

1. Introduction

Self-avoiding walks (SAW) are random walks with the restriction that no site of a lattice is visited more than once. From the physical point of view the flexible linear polymers are modelled as SAW, because of the excluded volume effect associated with linear polymers. For the last few years much attention has been paid to the study of the statistics of SAW on pure (Euclidean) lattices using approximate analytical theories and numerical methods. The main quantities of interest in the study of these statistics are the total number of configurations on the N-step walk (G_N) and their mean square end-to-end distance (R_N^2). These are respectively the zeroth and the second moment of the SAW distribution function $G_N(r)$, giving the number of SAW configurations of N steps with end-to-end distance r, and R_N^2 and G_N are found to fit the scaling forms (see, e.g., McKenzie 1976)

$$G_N\left(=\sum_r G_N(r)\right) \sim \mu^N N^{\gamma-1}$$

and (1.1)

$$R_N^2\left(= G_N^{-1}\sum_r r^2 G_N(r)\right) \sim N^{2\nu}.$$

μ is the connective constant and ν and γ are two critical exponents dependent on the lattice dimensionality d. Since the partition function graphs of an n-vector model in the limit $n \to 0$ are all self-avoiding graphs, the above exponents ν and γ are the

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correlation length and the susceptibility exponents, respectively, of the n-vector model in the limit $n \to 0$ (de Gennes 1972, 1979).

Recently, the effect of quenched randomness (in the lattice) on the critical behaviour of SAW statistics is being studied with great interest and these studies have apparently led to some confusion. It is well known that quenched configurational averaging over the random walk distribution can lead to a non-Markovian nature (see, e.g., Stauffer (1985) for the non-Markovian nature of random walk statistics on percolation clusters). SAW are non-Markovian to start with and configurational averaging on random (fractal) lattices is expected to lead to some interesting features. The problem was first studied by Chakrabarti and Kertész (1981) using a simple-minded application of the Harris criterion (Harris 1974) for disordered systems to the SAW equivalent n-vector magnetic model in the limit $n \to 0$. They found an indication of a change of the end-to-end distance exponent ν at weak concentration of dilution. However, a modified analysis (Harris 1983) of the Harris criterion in the n-vector model in the limit $n \to 0$ indicates that the critical behaviour of SAW statistics is not affected by lattice dilution even though the specific heat exponent α for SAW is positive. This has been partially supported by the field theoretic renormalisation group calculations (Kim 1983) where, however, an instability of the system is apparent when the system randomness grows beyond a certain limit, indicating, in such cases, a first order type transition. The real space renormalisation group (RSRG) studies by Roy and Chakrabarti (1982), Lam and Zhang (1984) and Sahimi (1984) also reflect the point that the critical behaviour of SAW will remain unchanged for weak dilution. Moreover, the phase diagram for SAW on random lattices was studied by Roy and Chakrabarti (1982) employing the RSRG method and compared well with that obtained by numerical methods (exact enumeration technique) (Chakrabarti *et al* 1983). Apart from these approximate analytical results, Monte Carlo simulations (Kremer 1981) and small series analysis (Hiley *et al* 1977) indicated similar observations, in that there is no change in the ν value of SAW by lattice dilution ($\nu^p = \nu$ for $p > p_c$), thus confirming the analytical results. Until now we made a brief review for the studies of the statistics of SAW on random lattices for $p > p_c$. We are now calling our attention to the study of SAW on percolation clusters at criticality, i.e. at $p = p_c$, where the dimension of the lattice is changed from Euclidean dimension d to some fractal dimension d_F. This study was first performed by Kremer (1981) employing the Monte Carlo simulation method on three-dimensional diamond lattices and observed that the exponent ν crosses over to a different ν^{p_c} at $p = p_c$. Kremer (1981) also suggested an approximate modified Flory type formula for $\nu^{p_c} = 3/(2 + d_F)$, which agrees well with the Monte Carlo simulation results. This ν^{p_c} value was also calculated employing a large cell RSRG method (Lam and Zhang 1984) and Monte Carlo RSRG method (Sahimi 1984) and was found to be in good agreement with Kremer's modified Flory formula. Recently Roy and Manna (1985) studied directed SAW (DSAW) on percolation clusters at $p = p_c$ by the large cell RSRG method and found a value for $\nu_\perp^{p_c}$ (the critical exponent for the end-to-end distance, perpendicular to the ballistic motion) in good agreement with the modified Alexander–Orbach formula ($\nu_\perp^{p_c} = d_S/2d_L$) (Roy and Manna 1985) where d_S is the spectral dimension and d_L is the spreading dimension of percolation clusters at $p = p_c$ ($\nu_\parallel^{p_c} = \nu_\parallel = 1$). It should be mentioned at this stage that all these studies were performed by direct disorder (configurational) averaging over the SAW distribution function $G_N(r)$.

Derrida (1982, 1984) argued that (since $G_N(r)$ is not Gaussian) the above direct averaging does not solve the problem completely as there will remain a difference between the mean value and the most probable value of the SAW distribution function.

He suggested that such a situation could be improved if one takes a configuration average over the logarithm of $G_N(r)$, which will make it a self-averaging quantity. In fact, the log average contains more physics of a disordered system, as this type of averaging is required to have information regarding any finite temperature thermodynamic quantity like the average elastic constant of a linear polymer in impure solvents, etc, where one has to average over the entropy (free energy) (Roy and Chakrabarti 1984). Recently Roy and Chakrabarti (1984) studied this type of averaging for DSAW (exactly) and also for ordinary SAW (enumeration) on diluted lattices and they found that the critical behaviour remains unchanged below a certain dilution concentration $c^* = 1 - p^*$, dependent on the length of the chains considered ($c^* \to 0$ for $N \to \infty$) and a crossover to a higher-order critical behaviour occurs beyond that point. In another paper, by Nadal and Vannimenus (1985), the model of DSAW on a diluted lattice has been studied with log averaging and it was shown that with any amount of disorder the mean value of the number of DSAW is different from its most probable value, agreeing with Derrida (1984). They also observed a crossover in γ at an amount of disorder, dependent on the chain length considered, in agreement with Roy and Chakrabarti (1984).

As such, there is a lot of confusion in the existing literature in comparing these results obtained using various kinds of configurational averaging. One should keep in mind that the results (e.g. $\nu^p = \nu$ for $p > p_c$ and $\nu^p = \nu^{p_c} \neq \nu$ for $p = p_c$; crossover at p_c) obtained using direct configurational averaging are not comparable, and should not be compared, with that using log average (e.g. $\gamma^p \neq \gamma$ for any $p < 1$; crossover at $c = 1 - p = 0$). Therefore, one should define the problem of SAW statistics on random lattices in a more refined way. SAW on random lattices involve two random processes—first we have to average over many SAW configurations of fixed number of steps (N) on the same lattice configuration of a particular dilution concentration and second we have to average over many lattice configurations of a particular dilution concentration. This second disorder averaging over the SAW configurations of given N is important. We observed from the previous studies that there exists broadly two types of disorder averaging in the problem—*direct disorder average over* $G_N(r)$ and *disorder average over logarithm of* $G_N(r)$. We also noticed that these two types of configurational averaging lead to different critical behaviours. Apart from the problem of disorder averaging, we have to consider where the self-avoiding walker will move on the disordered lattice. One can restrict the movement of SAW only on the infinite percolation cluster, and this we call *infinite cluster averaging*. On the other hand, one may choose a more liberal approach so that one can execute SAW on any cluster chosen randomly, which we call *all cluster averaging*.

In this paper, we will concentrate only on the direct disorder averaging over $G_N(r)$, for both types of SAW movements: only on the infinite cluster (§ 2) and on all clusters (§ 3). We have studied here a scaling function representation for R_N both for infinite cluster averaging (§ 2.1) and for all cluster averaging (§ 3.1). We have also proposed a scaling function representation of G_N for infinite cluster averaging (§ 2.4). G_N for all cluster averaging is trivial and is discussed in § 3.2.

2. SAW restricted to move on infinite cluster only (infinite cluster average)

In this section, we will try to formulate a scaling formalism for both the average end-to-end distance R_N and the total number of SAW configurations G_N on random

lattices, for the case where the SAW is executed only on the infinite percolation cluster. We also study the same, making use of the superlattice picture ('node-link-blob' model for $p > p_c$) of the infinite cluster.

2.1. Scaling theory for R_N

For infinite cluster averaging both Monte Carlo (Kremer 1981) and small series (Hiley *et al* 1977) studies as well as field theoretic arguments for the disordered n-vector model in the limit $n \to 0$ suggests (Kim 1983) that ν remains unchanged for any dilution above p_c. This is also supported by the extended Harris criterion (Harris 1983). However, since with increasing dilution, random folding of SAW chains become increasingly difficult, the magnitude of R_N increases with dilution (see, e.g., Kremer 1981) and we thus write for $p > p_c$

$$R_N \sim (p - p_c)^{\sigma} N^{\nu} \tag{2.1a}$$

with $\sigma \leqslant 0$. It may be mentioned that for a random walk on dilute lattices 2σ is the conductivity exponent t (>0) (see, e.g., Stauffer 1985). Here we do not have a comparable estimate of σ. For $p = p_c$, there are indications, from Monte Carlo simulation results (Kremer 1981) and RSRG results (Lam and Zhang 1984, Sahimi 1984), that there will be a crossover and the critical exponent ν will change to some ν^{p_c} value. This change in ν at p_c is also expected from the fact that the SAW critical exponents are dependent only on the dimensionality and the infinite percolation cluster at p_c will have dimensionality d_F different from the Euclidean dimensionality d. Thus, at $p = p_c$,

$$R_N \sim N^{\nu^{p_c}}. \tag{2.1b}$$

For p below p_c we have to restrict the motion of the SAW only on the 'incipient infinite cluster' and this will force the SAW, for some intermediate step sizes, to move on the boundary or the periphery of the incipient infinite cluster. Thus, obviously there will be no N dependence on R_N, and R_N will be of the order of the average size of the cluster for such N values. So for $p < p_c$,

$$R_N \sim \xi_P \sim (p_c - p)^{-\nu_P} \tag{2.1c}$$

where ξ_P is the percolation correlation length and ν_P is the corresponding exponent.

We now combine all results for R_N in three different limits of p into one scaling form:

$$R_N \sim N^{\nu^{p_c}} F[N^x (p - p_c)]. \tag{2.2}$$

For $p > p_c$ and for $Z \to \infty$ (where $Z \equiv N^x (p - p_c)$, $N \to \infty$) the scaling function $F(Z)$ must be proportional to Z^{σ} so as to be consistent with equation (2.1a) and thus $\sigma = (\nu - \nu^{p_c})/x$. Again, for $p < p_c$, $F(Z)$ should be proportional to $Z^{-\nu_P}$ so as to be consistent with equation (2.1c) and thus $x = \nu^{p_c}/\nu_P$. So we get

$$\sigma = (\nu_P / \nu^{p_c})(\nu - \nu^{p_c}). \tag{2.3}$$

Kremer (1981), from Monte Carlo simulation results, and Lam and Zhang (1984) and Sahimi (1984), from large cell and Monte Carlo RSRG studies, found ν^{p_c} to be greater than ν, ensuring therefore $\sigma \leqslant 0$. Kremer also found a similar expression for σ with a similar kind of argument for scaling function and the scaling form was also checked by a Monte Carlo simulation method (Kremer 1981).

2.2. Estimate of σ from the 'superlattice' model

We now use the de Gennes (1976) and Skal and Shklovskii (1975) 'nodes and links' model for the infinite percolation cluster at $p > p_c$ (superlattice picture) to study the statistics of SAW on infinite percolation clusters for $p > p_c$, and to have an independent estimate for the exponent σ in equation (2.3). The nodes of the superlattice are separated by a 'crow-flying' distance of the order of ξ_P and also by a chemical distance (the length of quasi-one-dimensional links) L_P where it is assumed that $L_P \sim (p - p_c)^{-\zeta_P}$. Since for $R_N \gg \xi_P$, the SAW will effectively see an Euclidean lattice, we assume the same relation as (1.1), for the size variation, with properly scaled variables: the end-to-end distance R_N as R_N/ξ_P and the number of steps N as N/L_P. We thus write

$$\frac{R_N}{\xi_P} \sim \left(\frac{N}{L_P}\right)^{\nu}$$

i.e.

$$R_N \sim (p - p_c)^{\zeta_P \nu - \nu_P} N^{\nu}$$

giving

$$\sigma = \zeta_P \nu - \nu_P \tag{2.4}$$

comparing with equation (2.1*a*). It is now well established, both analytically (Coniglio 1982) and numerically (Pike and Stanley 1981), that $\zeta_P = 1$ for the 'nodes and links' model. But it will be correct to compare σ of equation (2.3) with $(\zeta_P \nu - \nu_P)$ taking $\zeta_P = 1$ for $d \geqslant 6$ only, as the 'nodes and links' model for the infinite percolation cluster for $p > p_c$ is true for $d \geqslant 6$ and not for any dimension $d < 6$: the comparison is quite good for $d \geqslant 6$ as both σ and $(\zeta_P \nu - \nu_P)$ values are zero. For dimensions less than six, in the superlattice picture, the nodes are connected by links along with blobs ('nodes, links and blobs' model) (Stanley 1977, Coniglio 1982). According to Coniglio (1982), we can transform this 'nodes, links and blobs' picture into a 'nodes and links' picture by replacing the links and blobs in between two successive nodes with an effective one-dimensional chain (only link), provided that its length depends on the quantity under consideration. Here the physical quantity under consideration is SAW on diluted lattices and therefore the chemical path length L_P will be the average number of steps among the SAW between the extreme ends of the infinite cluster, and obviously $\zeta_P = \zeta_S$ where, as pointed out by Coniglio (1982), $\zeta_S = \nu_P / \nu^{P_c}$. Putting this value of ζ_P into (2.4) we see that this is exactly equal to σ of equation (2.3). Therefore one can say that the superlattice model is a good model to study SAW on random lattices for $p > p_c$. Such a method can also be applied to random walks on infinite percolation cluster ($p > p_c$) and the scaling relation for the conductivity exponent t compares well with the numerical data. In this case we can write $\zeta_{RW} = \nu_P / k$ where k is the end-to-end distance exponent of a random walk on the infinite cluster at $p = p_c$ and it can be shown that $t = (\nu_P / k)(1 - 2k) + \beta_P$, where β_P is the percolation order parameter exponent, and this is exactly the relation found from the scaling theory formalism (Gefen *et al* 1983, see also Stauffer 1985).

We now know that $\zeta_S = \nu_P / \nu^{P_c}$ and also that $\zeta_S = 1$ (exactly) (Coniglio 1982) for $d \geqslant 6$ and $\nu_P = \frac{1}{2}$ (Stauffer 1979). So for $d \geqslant 6$, $\nu^{P_c} = \frac{1}{2}$. We believe that this is an exact result for $d \geqslant 6$. This, at least for $d \geqslant 6$, actually gives support to both Kremer's (1981) modified Flory formula $\nu^{P_c} = 3/(2 + \bar{d}_F)$ and Rammal *et al* (1984) $\nu^{P_c} = (1/\bar{d}_B)[3\tilde{d}_B/(2 + \tilde{d}_B)]$ where $\bar{d}_B$ is the fractal dimension and $\tilde{d}_B$ is the spectral dimension

of the backbone of the infinite percolation cluster at $p = p_c$. But the Rammal *et al* (1984) formula for ν^{P_c} will give $\nu^{P_c} < \nu$ for $d < 6$, which is neither expected (SAW are expected to swell more with dilution, cf Sahimi (1984)) nor supported by the results of Monte Carlo simulation (Kremer 1981) or large cell (Lam and Zhang 1984) and Monte Carlo RSRG (Sahimi 1984). We thus think that Kremer's formula for ν^{P_c} might be the correct estimate of ν^{P_c} for $d < 6$.

2.3. *Phase diagram for SAW on random lattices near* p_c

One important piece of information which we are missing from the above scaling of R_N (equation (2.2)) is the nature of the phase diagram for SAW on random lattices, particularly near the percolation threshold p_c where the scaling holds good. To extract such an information we try to formulate the scaling in a slightly different way. Before that we redefine the end-to-end distance of SAW R_N and for this we switch over from a 'canonical ensemble' description to a 'grand canonical ensemble' description of the walks (Shapiro 1978, Family 1980). In the 'canonical ensemble', the average of R_N is taken over all SAW of the same number of steps ($R_N^2 = G_N^{-1} \Sigma_r r^2 G_N(r)$) whereas for the 'grand canonical ensemble' SAW of all step lengths N are allowed, with a weight f^N for an N-step walk in the ensemble. We then define the mean end-to-end distance of SAW (ξ_{SAW}) as $\xi_{SAW}^2(f) = \Sigma_r r^2 \Sigma_N G_N(r) f^N (\Sigma_r \Sigma_N G_N(r) f^N)^{-1}$ and we observe (using (1.1)) that at a critical fugacity f_c ($= 1/\mu$) $\xi_{SAW} \sim (f - f_c)^{-\nu}$.

It is by now clear that there is a crossover in the region $p \sim p_c$ and let the critical fugacity at p_c be $f_c(p_c)$. Let us assume that the crossover in the region $p \sim p_c$ can be expressed using a scaling form for ξ_{SAW}^{-1} as

$$\xi_{SAW}^{-1} \sim (p - p_c)^{\nu_P} \phi((f - f_c(p_c))^{\varphi} / (p - p_c)). \tag{2.5}$$

We know that $\xi_{SAW}^{-1} \sim (f - f_c(p_c))^{\nu^{P_c}}$ at $p = p_c$ (here f will act as the relevant field for the transition) and $\xi_{SAW}^{-1} \sim \xi_P^{-1} \sim (p - p_c)^{\nu_P}$ at $f = f_c(p_c)$ (here p will act as the relevant field for the transition). These two limiting forms can be reproduced from the above scaling form (2.5), provided we take the limiting forms for the function $\phi(X)$ (where $X \equiv (f - f_c(p_c))^{\varphi} / (p - p_c)$) as $\phi(X) \sim$ constant at $X \to 0$ and $\phi(X) \sim X^{\nu_P}$ at $X \to \infty$. Therefore, the crossover exponent is $\varphi = \nu^{P_c} / \nu_P$, which is the same as $1/x$ of § 2.1. But the main beauty of the scaling form is that the phase diagram for SAW on random lattices can be found out near p_c, because we know ξ_{SAW} will diverge at the critical points and the line joining the critical points for different p will give the phase diagram. ξ_{SAW} diverges where $\phi(X)$ has a zero (say at X_0) and therefore the critical fugacity at p is given by

$$f_c(p) = f_c(p_c) + \text{constant}(p - p_c)^{1/\varphi}. \tag{2.6}$$

Here, however, a point to note is that this formula will hold good only in the critical region of p_c because the scaling forms were constructed only to agree with the asymptotic forms of ξ_{SAW} in that region.

2.4. *Scaling for* G_N

Let us remind ourselves that we are continuing our discussion of infinite cluster averaging and we consider now the scaling for the total number of SAW G_N on random lattices. Using such a scaling picture, we show that the exponent γ will crossover to γ^{P_c} ($= \gamma + d(\nu^{P_c} - \nu)$) at $p = p_c$. However, one cannot rule out the possibility of

non-existence of γ^{p_c} at $p = p_c$, as one could possibly fit G_N to $\mu_{p_c}^N \exp(\sqrt{N})$ instead of $G_N \sim \mu_{p_c}^N N^{\gamma^{p_c}-1}$, where μ_{p_c} is the connective constant at p_c. Actually this kind of behaviour has been noticed in some cases of anisotropic excluded volume effects on SAW on pure lattices (Guttmann and Wallace 1985, Manna 1985) and has been searched for (with no conclusive evidence) for DSAW on diluted lattices (Nadal and Vannimenus 1985). Also, an exact solution (Kim and Kahng 1985) for G_N on a finitely ramified two-dimensional Sierpinski gasket (which has a fractal dimension $\ln 3/\ln 2$) gives a unique value of γ^{p_c} $(=2\ln[(13-3\sqrt{5})/6]/\ln[(7-\sqrt{5})/2])$, suggesting the existence of γ^{p_c} on random fractals (like percolation clusters). We also expect γ^{p_c} to be different from γ, as the infinite percolation cluster at p_c will have a dimension d_F different from the Euclidean dimension d and γ is dimension dependent.

Again we take N and $(p-p_c)$ as our scaling variables and a scaling function of the form $Y(N^x(p-p_c))$. We take x the same as before because the crossover is of the same kind and so x is known in this case $(x = \nu^{p_c}/\nu_P)$, and we write

$$G_N \sim \mu_p^N N^{\gamma^{p_c}-1} Y(N^x(p-p_c)) \tag{2.7}$$

where μ_p is the connective constant at dilution concentration $(1-p)$ and $\mu_p < \mu$. Let us assume, for $p > p_c$,

$$G_N \sim \mu_p^N N^{\gamma-1}(p-p_c)^{\tau} \tag{2.8a}$$

since lattice dilution will not change the critical behaviour of SAW for $p > p_c$. Also, at $p = p_c$,

$$G_N \sim \mu_{p_c}^N N^{\gamma^{p_c}-1}. \tag{2.8b}$$

The above two limiting forms of G_N can be reproduced from the scaling form (2.7) provided $Y(Z)$ (where $Z \equiv N^x(p-p_c)$) behaves as $Y(Z) \sim Z^{\tau}$ at $Z \to \infty$ and $Y(Z) \sim$ constant at $Z = 0$. Since $x = \nu^{p_c}/\nu_P$,

$$\tau = (\nu_P/\nu^{p_c})(\gamma - \gamma^{p_c}). \tag{2.9}$$

We have two unknown exponents here: τ and γ^{p_c}. So we try now to estimate τ from an independent method, so that we can evaluate γ^{p_c}. We see that the superlattice picture is again very useful to estimate τ. For this we consider the scaling picture of $G_N(r)$, which is

$$G_N(r) \sim \mu_p^N N^{\gamma-1-d\nu} h\left(\frac{r}{R_N}\right). \tag{2.10}$$

In the superlattice model we know, from equation (2.4), that R_N can be expressed as $R_N \sim \xi_P(N/L_P)^{\nu}$, and so

$$G_N(r) \sim \mu_p^N N^{\gamma-1-d\nu} h(r/N^{\nu}(p-p_c)^{\zeta_P\nu-\nu_P}).$$

Therefore

$$G_N = \sum_r G_N(r)$$

$$= \mu_p^N N^{\gamma-1}(p-p_c)^{d(\zeta_P\nu-\nu_P)} \qquad \text{for } p > p_c$$

giving

$$\tau = d(\zeta_P\nu - \nu_P) = (\nu_P/\nu^{p_c})(\gamma - \gamma^{p_c}).$$

Taking $\zeta_P = \zeta_S = \nu_P/\nu^{p_c}$ (Coniglio 1982), we obtain

$$\gamma^{p_c} = \gamma + d(\nu^{p_c} - \nu). \tag{2.11}$$

Table 1. The theoretically estimated values of ν^{P_c} ($=3/(2+d_F)$), $\tilde{\nu}^{P_c}$ (from equation (3.3)), γ^{P_c} (from equation (2.11)) and $\tilde{\gamma}^{P_c}$ ($=\gamma$, from equation (3.4)) are given for different dimensions, taking the percolation exponent values from the best estimates (Stauffer 1979, 1985).

d	2	3	4	5	6
ν	$\frac{3}{4}$	0.588	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
ν^{P_c}	0.770 (0.767)[a]	0.656 (0.65)[b]	0.57	0.52	$\frac{1}{2}$
$\tilde{\nu}^{P_c}$	0.73	0.49	0.29	0.19	0
γ	43/32	1.166	1	1	1
γ^{P_c}	1.384 (1.383)[c]	1.379 (1.302)[c]	1.27 (1.22)[c]	1.12 (1.05)[c]	1 (1)[c]
$\tilde{\gamma}^{P_c}$ $(=\gamma)$	43/42	1.166	1	1	1

[a] Large cell and Monte Carlo RSRG results (Lam and Zhang 1984, Sahimi 1984).
[b] Monte Carlo simulation results (Kremer 1981).
[c] From equation (2.13).

The values of γ^{P_c} for different dimensions are given in table 1. We notice that for $d \geq 6$ ($d_F = 4$), $\gamma^{P_c} = 1$ which should be obvious, as in these dimensions the SAW behave like random walks (the excluded volume effect disappears for dimensions above four). Moreover we know that $\gamma = 1$ reflects the difficulty for a SAW to return near its starting point (de Gennes 1979) and this is very true on the infinite percolation cluster at $p = p_c$ for $d \geq 6$, because of the disappearance of the blobs (Coniglio 1982) so that it becomes difficult for a SAW to return to its origin.

We can also find γ^{P_c} by an alternative method, making use of a scaling form for the 'grand canonical' generating function G ($=\Sigma_N G_N f^N$) of SAW. We can write, in a fashion similar to (2.5),

$$G \sim (p - p_c)^{-\gamma_P} \psi[(f - f_c(p_c))^{\tilde{\varphi}}/(p - p_c)]. \tag{2.12}$$

We know that $G \sim (f - f_c(p_c))^{-\gamma^{P_c}}$ at $p = p_c$ and $G \sim (p - p_c)^{-\gamma_P}$ at $f = f_c(p_c)$ where γ_P is the susceptibility exponent for percolation. These two limiting forms are reproduced from the scaling equation (2.12) provided we take the form of the scaling function $\psi(U)$ (where $U \equiv (f - f_c(p_c))^{\tilde{\varphi}}/(p - p_c)$) as $\psi(U) \sim$ constant at $U \to 0$ and as $\psi(U) \sim U^{\gamma_P}$ at $U \to \infty$. We thus get the crossover exponent as $\tilde{\varphi} = \gamma^{P_c}/\gamma_P$. However, since the crossover is of the same kind (because of the singularities at the percolation point), we may write $\tilde{\varphi} = \varphi$ ($=\nu^{P_c}/\nu_P$) and so we obtain

$$\gamma^{P_c} = \frac{\nu^{P_c}}{\nu_P} \gamma_P. \tag{2.13}$$

The estimated values of γ^{P_c} from this equation is also given in table 1 and compared with those obtained from equation (2.11).

3. SAW on any cluster chosen randomly (all cluster averaging)

In this section we will confine ourselves to the study of the statistics of SAW on random lattices where the SAW can be executed on any cluster chosen randomly. For this, we have to perform all cluster averaging for the disorder. We develop here a scaling

formalism for the end-to-end distance of the SAW, similar to that of random walks on any cluster of a disordered lattice (Gefen *et al* 1983). We also make here a comment on the average number of SAW configurations for all cluster averaging.

3.1. Scaling for R_N

We try first to write the expressions for R_N in three different limits of p.

For $p > p_c$

$$R_N \sim (p - p_c)^{\tilde{\sigma}} N^{\nu}. \tag{3.1a}$$

ν is here again the same as that for SAW on a pure lattice for the same reason as that for the infinite cluster averaging, discussed before. However, $\tilde{\sigma}$ is different from σ of equation (2.1*a*); the factor $(p - p_c)^{\tilde{\sigma}}$ in (3.1*a*) can be replaced (see, e.g., Stauffer 1985) by $(p - p_c)^{\sigma}(P/p)$, where P is the percolation order parameter, giving $\tilde{\sigma} = \sigma + \beta_P/2$, β_P being the percolation order parameter exponent. It may be noted that for annealed impurities, all cluster averaging (Harris 1983) gives $\tilde{\sigma} = 0$ because of the validity of the independent configurational averaging in the numerator and the denominator in R_N in (1.1) (cf Rammal *et al* 1984, Lyklema and Kremer 1984).

At $p = p_c$

$$R_N \sim N^{\tilde{\nu}^{p_c}} \tag{3.1b}$$

where $\tilde{\nu}^{p_c}$ is not equal to ν^{p_c}, the critical end-to-end distance exponent of SAW when the movement of the SAW is restricted only on the infinite cluster. Similar to the random walk problem on any cluster (cf Stauffer 1985), we may write for $p < p_c$

$$R_N^2 = \sum_S P_S R_S^2 \sim (p - p_c)^{\beta_P - 2\nu_P}$$

for some finite values of N, where the average over all clusters of different masses S is reflected through R_S, the average cluster radius of size S, and P_s is the probability of a site belonging to a cluster of mass S. Thus, for $p < p_c$,

$$R_N \sim (p - p_c)^{\beta_P/2 - \nu_P}. \tag{3.1c}$$

Knowing these three limits, we may combine them into one scaling form with scaling fields N and $(p - p_c)$ as usual. Let us have a scaling function $\theta(N^y(p - p_c))$, where y is an unknown crossover exponent, and we write

$$R_N \sim N^{\tilde{\nu}^{p_c}} \theta(N^y(p - p_c)). \tag{3.2}$$

Reproducing the three limits from this scaling equation, we get the crossover exponent as $y = \tilde{\nu}^{p_c}/(\nu_P - \beta_P/2)$ and $\tilde{\nu}^{p_c} = \nu(\nu_P - \beta_P/2)/(\nu_P + \sigma)$. Knowing σ from equation (2.3) of the previous section, we obtain

$$\tilde{\nu}^{p_c} = \nu^{p_c}(1 - \beta_P/2\nu_P). \tag{3.3}$$

The values of $\tilde{\nu}^{p_c}$ for different dimensions are given in table 1.

3.2. Average G_N

In fact, for all cluster averaging, γ can be shown to remain unchanged even at $p = p_c$,

i.e. $\tilde{\gamma}^{p_c} = \gamma$. This is because

$$G_N = \sum_{\{C\}} P_{\text{walk}}$$

where $\{C\}$ = all the SAW configuration on the pure lattice and $P_{\text{walk}} = 1$ if the SAW configuration is possible on the dilute lattice and is zero otherwise. As such, $P_{\text{walk}} = p_1 p_2 p_3 \ldots p_N$, where p_i are the occupation operators ($p_i = 1$ or 0) of the bonds of the lattice through which the SAW configuration on the pure lattice passes. Now, since a SAW never visits any bond more than once, all p_i are different and

$$\overline{\prod_i p_i} = (\overline{p_i})^N = p^N$$

giving

$$G_N \equiv \overline{G_N} = p^N \sum_{\{C\}} 1 = \mu_p^N N^{\gamma - 1} \tag{3.4}$$

with $\mu_p = p\mu$. γ, thus, remains exactly the same as that on the pure lattice (Harris 1983, Lyklema and Kremer 1984, Rexakis and Argyrakis 1983).

4. Summary

In summary, we have studied SAW on lattices having quenched random dilution where the SAW are restricted to move (a) only on the infinite percolation cluster and (b) on any cluster chosen randomly. The disorder (configurational) averaging procedure we employed is the direct average over $G_N(r)$. We have studied a scaling function representation for R_N, which describes a crossover from ordinary SAW (for $p > p_c$) to SAW on a percolation cluster at $p = p_c$, both for infinite and all cluster averagings. For all cluster averaging we find the end-to-end distance exponent $\tilde{\nu}^{p_c}$, of SAW on a percolation cluster at $p = p_c$, to be related to the end-to-end distance exponent ν^{p_c} for SAW restricted on the infinite percolation cluster: $\tilde{\nu}^{p_c} = \nu^{p_c}(1 - \beta_P/2\nu_P)$ where $\nu^{p_c} \simeq 3/(2 + d_F)$. We have also shown that a scaling function representation of G_N indicates that γ will change to γ^{p_c} at $p = p_c$ where $\gamma^{p_c} = \gamma + d(\nu^{p_c} - \nu)$ for infinite cluster averaging where, as for all cluster averaging, γ will remain unchanged even at $p = p_c$.

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Note added in proof. It may be noted that there will be no self-avoiding polygons (SAP) for walks on infinite percolation cluster at $p = p_c$ ($C_\curlywedge = 0$ for $N \to \infty$, where $C_\curlywedge$ is the total number of SAP). This is essentially because of the (singly connected) links in the infinite percolation cluster structure at p_c. Therefore, $\mu_{p_c}^L$ (giving $C_\curlywedge \sim (\mu^L)^\curlywedge N^{\alpha - 2}$) for SAP is zero at p_c and thus is definitely different from μ_{p_c} for SAW. (This may be true for $p \geq p_c$ also.) Thus, μ^L (for loops) $< \mu$ (for walks) on fractal lattices and this has also been observed for some constrained (e.g. directed) SAW on Euclidean lattices (Manna 1986).

In § 2.4, we discussed that there might have a possibility of non-existence of γ for SAW on (random) fractal lattices. Recently Vannimenus and Knežević (1986) have shown exactly, following the original study of SAW statistics on non-random fractals by Dhar (1978), the non-existence of γ for branched polymers on Sierpinski type (non-random) fractals, although for SAW on such fractals γ seems to be clearly defined (Elezovic *et al* 1986).

References

Chakrabarti B K, Bhadra K, Roy A K and Karmakar S N 1983 *Phys. Lett.* **93A** 434
Chakrabarti B K and Kertész J 1981 *Z. Phys.* B **44** 221
Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** 3829
de Gennes P G 1972 *Phys. Lett.* **38A** 339
—— 1976 *J. Physique Lett.* **37** L1
—— 1979 *Scaling Concepts in Polymer Physics* (Ithaca, NY: Cornell University Press)
Derrida B 1982 *J. Phys. A: Math. Gen.* **15** L119
—— 1984 *Phys. Rep.* **103** 29
Dhar D 1978 *J. Math. Phys.* **19** 5
Elezovic S, Knežević M and Milosevic S 1986 *Preprint* Belgrade
Family F 1980 *J. Phys. A: Math. Gen.* **13** L325
Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
Guttmann A J and Wallace K J 1985 *J. Phys. A: Math. Gen.* **18** L1049
Harris A B 1974 *J. Phys. C: Solid State Phys.* 7 1671
—— 1983 *Z. Phys.* B **49** 347
Hiley B J, Finney J L and Burke T 1977 *J. Phys. A: Math. Gen.* **10** 197
Kim D and Kahng B 1985 *Phys. Rev.* A **31** 1193
Kim Y 1983 *J. Phys. C: Solid State Phys.* **16** 1345
Kremer K 1981 *Z. Phys.* B **45** 149
Lam P M and Zhang Z Q 1984 *Z. Phys.* B **56** 155
Lyklema J W and Kremer K 1984 *Z. Phys.* B **55** 41
Manna S S 1985 *J. Phys. A: Math. Gen.* **17** L899
—— 1986 *Preprint* Saha Institute
McKenzie D S 1976 *Phys. Rep.* **27C** 25
Nadal J P and Vannimenus J 1985 *J. Physique* **46** 17
Pike R and Stanley H E 1981 *J. Phys. A: Math. Gen.* **14** L169
Rammal R, Toulouse G and Vannimenus J 1984 *J. Physique* **45** 389
Rexakis J and Argyrakis P 1983 *Phys. Rev.* B **28** 5323
Roy A K and Chakrabarti B K 1982 *Phys. Lett.* **91A** 393
—— 1984 *Z. Phys.* B **55** 131
Roy A K and Manna S S 1985 *Z. Phys.* B **61** 205
Sahimi M 1984 *J. Phys. A: Math. Gen.* **17** L379
Shapiro B 1978 *J. Phys. C: Solid State Phys.* **11** 2829
Skal A and Shklovskii B I 1975 *Sov. Phys.-Semicond.* **8** 1029
Stanley H E 1977 *J. Phys. A: Math. Gen.* **10** L211
Stauffer D 1979 *Phys. Rep.* **54** 1
—— 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
Vannimenus J and Knežević M 1986 *Preprint* L'Ecole Normale Supérieure